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Undoing Orbifold Quivers

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Abstract

A number of new papers have greatly elucidated the derivation of quiver gauge theories from D-branes at a singularity. A complete story has now been developed for the total space of the canonical line bundle over a smooth Fano 2-fold. In the context of the AdS/CFT conjecture, this corresponds to eight of the ten regular Sasaki-Einstein 5-folds. Interestingly, the two remaining spaces are among the earliest examples, the sphere and T^{11}. I show how to obtain the (well-known) quivers for these theories by interpreting the canonical line bundle as the resolution of an orbifold using the McKay correspondence. I then obtain the correct quivers by undoing the orbifold. I also conjecture, in general, an autoequivalence that implements the orbifold group action on the derived cateory. This yields a new order two autoequivalence for the $\mathbb{Z}_2$ quotient of the conifold.

1. Introduction

The nature of singularities has long been one of the central interesting questions in string theory. One of the best techniques we have for understanding this has been to probe them with D-branes. Thus, it is of great interest to understand the gauge theory that lives on a D-brane situated at a singularity. This is particularly important in the context of the AdS/CFT conjecture wherein the most interesting gauge theories are obtained in such a manner.

While there have been a number of techniques used to derive such gauge theories, the most powerful at present seems to be that of exceptional collections in derived categories. First referenced in [1] and initially developed by [2], this technique has been used to derive quivers for the gauge theories resulting from all collapsing del Pezzo surfaces in Calabi-Yaus. The procedure in those references has now been substantially elucidated in the works of [3,4,5]. In particular, the last reference makes concrete most of the relevant mathematics. While the issues of superpotentials is still not completely worked out, we will see that in many cases we can use physical insight to guess the correct answer.

In the AdS/CFT conjecture, we take a stack of D3-branes on $\mathbb{R}^4$ located at the tip of a 6 (real-)dimensional cone with a Calabi-Yau metric of the form

$$ds^2 = dr^2 + r^2 ds^2_{M^5} \ . \tag{1.1}$$

When we take the near horizon limit of such a metric, it is easy to see that the resulting geometry is $AdS_5 \times M^5$. An interesting feature of the metric (1.1) is that the radial coordinate r does not have to extend to zero. Any CY metric of this form defines a Sasaki-Einstein (SE) structure on M^5. A characteristic feature of such metrics is that they have a nondegenerate vector field that can be integrated to a foliation of the 5-fold. These foliations can be divided into three different types: (1) *regular*, where all the leaves have the same lengths; (2) *quasiregular*, where the leaves can have different lengths and (3) *irregular*, where the leaves are noncompact. In case (1), the space of leaves is a Fano[1] Kähler-Einstein (KE) 2-fold, and in case (2) it is a Fano Kähler-Einstein orbifold. The last case remains mysterious, although some examples have been recently investigated [6,7,8,9,10,11,12,13,14]. For the remainder of this paper, we will deal only with the first case.

In both the first two cases, the 5-fold is the total space of a circle (V-)bundle over the base. All smooth Fano, KE 2-folds are known. They are the third through eighth del Pezzo surfaces[2], $dP_3, \ldots, dP_8$, $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$. All regular SE 5-folds are circle bundles over these spaces. It was shown by Friedrich and Kath [15] that the Euler class of these must integrally divide the canonical class of the base. This gives the following complete classification. First, we introduce the *Fano index* of a Fano variety, X. This is the largest natural number, I, such that $c_1(X)/I \in H^2(X, \mathbb{Z})$. The Fano index of all the del Pezzos is 1, so the only SE space is the total space of the circle bundle corresponding to the canonical class. For $\mathbf{P}^2$, the Fano index is 3, so we have circle bundles corresponding to the canonical class and the canonical class divided by three. These manifolds are easily seen to be $\mathbf{S}^5/\mathbb{Z}_3$ and $\mathbf{S}^5$ respectively. Finally, the Fano index of $\mathbf{P}^1 \times \mathbf{P}^1$ is 2 and the corresponding 5-folds are denoted $T^{11}/\mathbb{Z}_2$

[1] Fano means that the the anticanonical line bundle is ample (positive).

[2] The notation dP_n refers to $\mathbf{P}^2$ blown up at n points.

and T^{11}. In all these situations, the cone given by the metric (1.1) can be thought of as 'filling in' the circle bundle, *i.e.,* as the total space of the line bundle with same first Chern class as the Euler class of the circle bundle. As noted earlier, the metric may not extend to the zero section of this line bundle.

While all these SE 5-folds are relevant for the AdS/CFT conjecture, the techniques given in [3,4,5] only work for the case of the canonical bundle. The reason for this is that, in order to work at large volume, we must be able to blow up the base of the cone. However, the CY metric only extends to the zero section when the canonical class of the cone is trivial which implies that the bundle must be the total space of the canonical bundle of the base. However, all hope is not lost for the other two cases. In each case, the singularity we obtain when we collapse the 'tip' of the cone to a point has an alternate resolution. For T^{11}, the cone is the conifold with its well-known resolution to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$. For S^5, the cone is just $\mathbb{C}^3$ which needs no resolution.

While it is possible to analyze these two cases in terms of their CY resolutions, we will go a different route in this paper in the hope that the techniques developed will eventually be applicable to the less well-understood quasiregular case. We will exploit the fact that the canonical *circle* bundle is always an orbifold when the Fano index is greater than one. This suggests that we can interpret the quivers obtained by the techniques of [3,4,5] as orbifolds of the quivers we desire. We will show how this can be accomplished and show that it obtains the correct answers for $\mathbf{S}^5$ and T^{11}.

The paper is organized as follows. In section 2, we introduce exceptional collections and helices and show how to obtain quiver gauge theories from them. We make a conjecture as to the form of the superpotential. In section 3, we present an algorithm to undo the orbifold of the quiver, and we show that it gives the correct results. In section 4, we use the simpler example of $\mathbb{C}^*$ bundles to motivate the algorithm and to introduce some orbifold techniques. In section 5, we show how we can use the McKay correspondence to treat the derived category of coherent sheaves on the total space of the canonical bundle as an equivariant derived category. In section 6, we conjecture a Fourier-Mukai transform that implements the action of the orbifold group on the derived category and show how this justifies the algorithm presented in section 3.

2. Exceptional collections, helices and quivers

We will assume familiarity with the language of derived categories. Excellent references on the subject are Aspinwall's review [16] and the textbooks [17,18]. Because exceptional collections are well established in this area, we will be brief in our review. For more details, see [3,4] and references therein.

2.1. Exceptional collections and quivers

An exceptional object, E, in a derived category[3] satisfies the following identities:

$$\begin{aligned} \mathrm{Hom}(E,E) &= \mathbb{C} \ , \\ \mathrm{Hom}(E,E[k]) = \mathrm{Ext}^k(E,E) &= 0 \quad \text{for} \quad k \neq 0 \ . \end{aligned} \tag{2.1}$$

An exceptional collection is an ordered collection of exceptional objects, E_i, $i = 0 \ldots n-1$, such that

$$i > j \Rightarrow \mathrm{Ext}^k(E_i, E_j) = 0 \quad \forall k \in \mathbb{Z} \ . \tag{2.2}$$

An exceptional collection is called *full* if it generates the triangulated category and is called *strong* if

$$\mathrm{Ext}^k(E_i, E_j) = 0 \quad \forall k \neq 0 \quad \forall i,j \ . \tag{2.3}$$

It was shown by Bondal [19] that, given the data of an full, strong exceptional collection of objects in a triangulated category, we can construct a quiver such that the derived category of representations of the quiver is equivalent to the original triangulated category. For our purposes, the original triangulated category will always be the derived category of coherent sheaves on some variety, X. Thus, we have an equivalence of triangulated categories $\mathcal{D}(X) \cong \mathcal{D}(A - \mathrm{Mod})$ where we identify representations of the quiver with modules of the quiver algebra, A.

The construction proceeds as follows. Given a full, strong exceptional collection, let us form the object

$$T = \bigoplus_{i=0}^{n-1} E_i \ . \tag{2.4}$$

Then, the endomorphism algebra of this object is the algebra of a quiver with relations. In fact, it will be more useful to look at the opposite algebra[4], so we define

$$A^{\mathrm{op}} = \mathrm{Hom}(T,T) \ . \tag{2.5}$$

We will assume familiarity with the theory of representations of quivers.[5] Recall that there are two types of distinguished representations of a quiver associated to any given node. First, there is the simple representation, S_i, given by a one dimensional vector space at that node with all maps equal to zero. Second, there is the projective representation, P_i, where, associated to the node labelled j, we have a vector space with basis given by the set of paths from node i to node j. The maps are given by concatenation. With these definitions, we have the identification

$$\mathrm{Hom}(E_i, E_j) \cong \mathrm{Hom}(P_i, P_j) = \text{paths from } j \text{ to } i \ . \tag{2.6}$$

[3] Really, any triangulated category.
[4] This essentially interchanges the roles of left and right modules.
[5] For an extensive reference, see [20]. For the relevant parts, see [4].

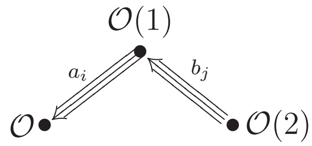

Figure 1: The quiver for $\mathbf{P}^2$.

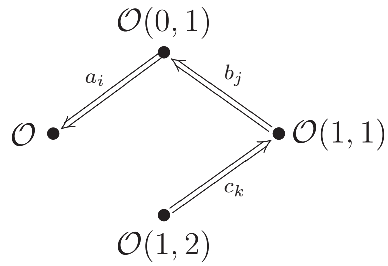

Figure 2: The quiver for $\mathbf{P}^1 \times \mathbf{P}^1$.

Let us now work out the quivers in the two examples that will be the focus of this paper. First, for $\mathbf{P}^2$, a full, strong exceptional collection is $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$. It is easy to compute

$$\begin{gathered} \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) = \mathbb{C}^3 \ , \\ \mathrm{Hom}(\mathcal{O}, \mathcal{O}(2)) = \mathbb{C}^6 \ . \end{gathered} \tag{2.7}$$

This tells us that the quiver is the one given in figure 1. To determine the relations, we note that the maps in equation (2.7) are simply multiplication by sections of $\mathcal{O}(1)$. The relations come from the fact that this multiplication is commutative, *i.e.*, $xy = yx \in H^0(\mathcal{O}(2))$ for $x, y \in H^0(\mathcal{O}(1))$. Thus, using the labels in the diagram, we have the relations $a_i b_j = a_j b_i$, $i, j = 1, 2, 3$. This can be succinctly written as $\epsilon^{ijk} a_j b_k = 0$.

In the case of $\mathbf{P}^1 \times \mathbf{P}^1$, we will work with the full, strong exceptional collection

$$(\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,1), \mathcal{O}(1,2)) \ . \tag{2.8}$$

Here, $\mathcal{O}(a,b)$ denotes the sheaf $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$. It is a short exercise to verify that the quiver is given as in figure 2 with the relations $a_i b_j c_k = a_k b_j c_i$, $i, j, k = 1, 2$. We can summarize this as $\epsilon^{ik} a_i b_j c_k = 0$.

2.2. Completing the quiver

Now that we have quivers corresponding to the 2-folds, we need to understand how to obtain quivers corresponding to the full cones. In particular, we will work with the total space of the canonical line bundle, denoted by K_X for the 2-fold X. Following Aspinwall [4], we will describe a somewhat *ad hoc* procedure for obtaining the correct quiver. Please see the original paper for more details.

We will show the next section how to obtain a collection of objects in $\mathcal{D}(X)$, S_i, dual to the exceptional collection E_i. These objects correspond to the simple representations of the

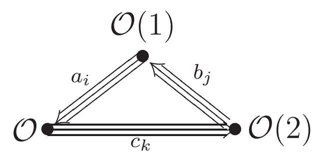

Figure 3: The completed quiver for $\mathbf{P}^2$.

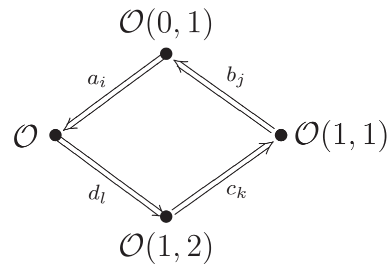

Figure 4: The completed quiver for $\mathbf{P}^1 \times \mathbf{P}^1$.

quiver. Some useful properties of these representations (see, for example, [4]) are

$$\begin{aligned} \dim\, \mathrm{Ext}^1(S_i, S_j) &= n_{ij} \\ \dim\, \mathrm{Ext}^2(S_i, S_j) &= r_{ij} \\ \dim\, \mathrm{Ext}^3(S_i, S_j) &= rr_{ij} \\ &\vdots \end{aligned} \quad . \tag{2.9}$$

Here, n_{ij} is the number of arrows (*not* paths) from node i to node j in the quiver, r_{ij} is the number of relations between paths that go from node i to node j and rr_{ij} is the number of relations between relations, *ad infinitum*.

Tensoring with one of these representations adds one to the dimension of the vector space at a given node. In the gauge theory, this corresponds to adding one to the rank of the gauge group at the node. This allows us to identify these representations with the fractional branes in the string theory. We only must embed them in the cone K_X to obtain the correct answer. Let s denote the zero section of the bundle K_X. Then, the relevant objects are $s_*(S_i)$.

There exists a spectral sequence that gives us the Ext groups between these objects [4,21,22]. It gives

$$\mathrm{Ext}^i_{K_X}(s_*(S_i), s_*(S_j)) = \mathrm{Ext}^i_X(S_i, S_j) \oplus \mathrm{Ext}^{3-i}_X(S_j, S_i) \ . \tag{2.10}$$

In order to avoid tachyons, we have to work with quivers that do not have any Ext^3s. What this equation then tells us is that, for any relation in the quiver for X, we must draw a line in the opposite direction in the quiver for K_X. We will call this new quiver the 'completed quiver'. Also, for any arrow in the original quiver, we have a relation in the completed quiver going in the opposite direction. Finally, we have relations between the relations at every node. The completed quivers for $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ are given in figures 3 and 4.

It now remains to determine the relations of the completed quiver. To do this rigorously is difficult, possibly involving higher products in the algebra of Ext groups.[6] We can easily

[6]Some computations of superpotentials in this context appear in [23].

guess a set of relations that satisfy the aforementioned properties, however. The key is to remember that relations in quiver gauge theories correspond to F-terms in the gauge theory and as such are given as the derivatives of a superpotential. Let us write the set of relations of the quiver for X as $R^a = 0$ where a ranges from 1 to $\sum r_{ij}$. Equation (2.10) tells us that, for each of these relations, we add a new arrow to the completed quiver which we will denote r_a. Note that any relation is a sum of paths from a node i to a node j while the corresponding added arrow points from j to i. Thus, the quantity

$$W = \sum_a r_a R^a \tag{2.11}$$

is gauge invariant in the quiver gauge theory. This is the superpotential, and the relations for the completed quiver are derived from it as

$$\frac{\partial W}{\partial a_i} = 0 \tag{2.12}$$

where a_i ranges over all arrows in the completed quiver. It is easy to see that this set of relations obeys all the properties implied by the equation (2.10).

Applying this to our examples, we obtain for $\mathbf{P}^2$:

$$W = \epsilon^{ijk} a_i b_j c_k \ , \tag{2.13}$$

and for $\mathbf{P}^1 \times \mathbf{P}^1$:

$$W = \epsilon^{ik} \epsilon^{lj} a_l b_i c_j d_k \ . \tag{2.14}$$

These are the well-known, correct superpotentials for these examples.

2.3. Mutations and helices

In order to justify the preceding manipulations and to identify the algebra of the completed quiver, we must first introduce the notions of a mutation and a helix.

There are in fact two notions of a mutation that we will need. We will begin by describing a mutation in a triangulated category. Given two objects, E and F, we define the left mutation, $L_E F$, by the triangle

$$L_E F \longrightarrow \mathrm{Hom}(E, F) \otimes E \longrightarrow F \tag{2.15}$$

where the second arrow is the evaluation map. It is not hard to see that, given an exceptional pair, (E, F), the pair, $(L_E F, E)$, is also exceptional. This defines a braid group action on exceptional collections [24]. We can similarly define a right mutation, but we will not need it here.

In fact, if the objects E and F are both the images of coherent sheaves in the derived category, the object $L_E F$ will often have a single nonzero cohomology sheaf. This can be proven to be the case[7] when our variety has no rigid torsion sheaves and $h^0(-K_X) \geq 2$. As

[7] This is a sufficient but not necessary condition.

this will always be the case here, let us define a new object $L^s_E F$, the left sheaf mutation, to be this cohomology sheaf. A more proper definition is given in [25].

We can now define the dual objects mentioned in the previous section. Given an exceptional collection $(E_0, \ldots, E_n)$, we can define the new collection

$$(F_0, \ldots, F_n) = (L^n E_n, L^{n-1} E_{n-1}, \ldots, L_{E_0} E_1, E_0) \ . \tag{2.16}$$

The notation L^n refers to n applications of left mutation (*not* sheaf mutations). For example, $L^2 E_2 = L_{E_0} L_{E_1} E_2$. These obey [5,19]

$$\mathrm{Ext}^k(E_i, F_{n-j}[j]) = \mathbb{C}\delta_{ij}\delta_{k0} \ . \tag{2.17}$$

This gives us the dual collection[8], $S_j = F_{n-j}[j]$. For the collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ over $\mathbf{P}^2$, the dual collection, S_i, is given by $(\Omega^0, \Omega^1(1)[1], \Omega^2(2)[2])$. For the collection (2.8), the dual collection is $(\mathcal{O}, \mathcal{O}(-1,0)[1], \mathcal{O}(1,-1)[1], \mathcal{O}(0,-1)[2])$.

Also, given an exceptional collection $(E_0, \ldots, E_n)$, we can take the rightmost element E_n and mutate it to the far left giving the new exceptional collection $(E_{-1}, \ldots, E_{n-1})$. It is a theorem of Bondal [19] that, if the exceptional collection is full, *i.e.,* generates the derived category, then

$$E_{-1} = L^n E_n = E_n \otimes K[m-n] \tag{2.18}$$

where m is the dimension of the variety that we are working on.

Bondal [19] defines a helix as a collection of objects, E_i, $i = -\infty, \ldots, \infty$, such that

$$E_i = E_{i+n} \otimes K[m-n] \ . \tag{2.19}$$

In particular, this defines a helix of length $n+1$. The relation (2.18) shows that any full exceptional collection is the basis of a helix that can be formed by left (and right) mutations.

This, however, does not seem to be the most appropriate definition of a helix.[9] In fact, most of the theorems about helices are proven [26] in the case $m = n$ in the above notation. As we will see, a better notion of a helix is a collection of *sheaves*, rather than objects in the derived category, that obey

$$E_i = E_{i+n} \otimes K \ . \tag{2.20}$$

Given an exceptional collection of length $n+1$, we can use the notion of sheaf mutation to define one of these helices by $E_{-1} = (L^s)^n E_n$ and the corresponding relation for right sheaf mutations. We will use the term helix solely to refer to a collection of sheaves that obey (2.20). This sort of helix appears, for example, in [25].

Bondal and Polishchuk [26] call a helix geometric if, for all $i \leq j$, it obeys

$$\mathrm{Hom}(E_i, E_j[k]) = 0 \text{ unless } k = 0 \ . \tag{2.21}$$

Bridgeland [5] calls these helices *simple*, and we will follow his usage. Note that, while we are applying Bondal's definition to our modified version of a helix, it still makes sense. In

[8] In most of the literature, the Fs are referred to as the dual collection. However, the Ss are the important objects for us, so we prefer to reserve the term dual for them.

[9] In this context, this fact has also been observed by Christopher Herzog and Tom Bridgeland to my knowledge.

fact, Bondal and Polishchuk [26] show that, for their definition of a helix, it can only be geometric/simple in the case that $m = n$, *i.e.,* when it is also a helix under our definition.

For our two running examples, on $\mathbf{P}^2$, we have the simple helix $E_i = \mathcal{O}(i)$ and on $\mathbf{P}^1 \times \mathbf{P}^1$

$$\ldots, \mathcal{O}(-1,-1), \mathcal{O}(-1,0), \mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,1), \mathcal{O}(1,2), \mathcal{O}(2,2), \mathcal{O}(2,3), \ldots . \tag{2.22}$$

2.4. Deriving the completed quiver

With these tools in hand, we can proceed to find a quiver algebra whose derived category of modules is equivalent to the derived category of the total space K_X. This material is from Bridgeland's paper [5].

Given a simple helix E_i generated by an exceptional collection $(E_0, \ldots, E_{n-1})$, we define the following graded algebra:

$$\bigoplus_{k \geq 0} \prod_{j-i=k} \mathrm{Hom}(E_i, E_j) \ . \tag{2.23}$$

which Bridgeland calls the *helix algebra*.[10] There is a natural $\mathbb{Z}$-action given by the isomorphism

$$\otimes K_X : \mathrm{Hom}(E_i, E_j) \longrightarrow (E_{i-n}, E_{j-n}) \ . \tag{2.24}$$

The invariant subalgebra under this action is called the *rolled-up helix algebra*. We will denote its opposite algebra by B[11].

Now, similar to the object T in (2.4), we can define on K_X:

$$\widetilde{T} = \bigoplus_{i=0}^{n-1} \pi^* E_i \tag{2.25}$$

where π is the projection from K_X to X. Proposition 4.1 of Bridgeland [5] states that the functor

$$\mathrm{Hom}(\widetilde{T}, \cdot) : \mathcal{D}^b(K_X) \longrightarrow \mathcal{D}^b(B - \mathrm{Mod}) \tag{2.26}$$

is an equivalence of triangulated categories.

A crucial element of the proof of this theorem is that, for any line bundle L over a space X with projection π, we have

$$\pi_*(\mathcal{O}_L) = \bigoplus_{p \leq 0} L^p \tag{2.27}$$

where we also denote by L the sheaf of sections of the line bundle L.

The degree zero part of B contains the idempotents $e_i = \prod_{k \in \mathbb{Z}} \mathrm{id}_{E_{i+nk}}$. Thus, we can define the projective modules $P_i = Be_i$ which are the images of the objects $\pi^* E_i$. There are also simple modules T_i such that $\dim(e_i T_j) = \delta_{ij}$. If we let s be the zero section of K_X, then the object $s_*(S_j) = s_*(F_{n-1-j}[j])$ is mapped to the T_j.

[10] This is similar to the helix algebra in [26].

[11] This is the opposite of Bridgeland's definition, but is consistent with our earlier definition of A and with the conventions of other papers.

Given the equivalence of categories, it is clear by the spectral sequence of the previous section that the quivers derived there are the quivers from this algebra. It would be interesting to use this to prove the conjecture about the superpotential from the previous section.

3. Undoing the orbifold

Now that we see how to derive the quiver for the total space K_X, we would like to interpret it as an orbifold and undo that orbifold. In this section, we will show how to do so algorithmically. We will somewhat justify these manipulations in proceeding sections.

Let us assume we are on a 2-fold, X, with Fano index I. Let us denote by k the line bundle such that $k^I = K_X$. The equivalence (2.26) of the previous section follows from the fact (2.27) which holds for any line bundle, L, and the fact that the action of tensoring with K_X preserves the helix. As we would like to describe the total space of the line bundle k, we would like to have a helix that is preserved by tensoring with k. If this were true, it would follow from the arguments in [5] that the invariant portion of the helix algebra under this action would have a derived category of modules equivalent to the derived category of coherent sheaves on the total space of k.

We now make the following:

Conjecture 1 *Given a Fano Kähler-Einstein 2-fold, X, with Fano index, I, and k such that $k^I = K_X$, there always exists a helix, E_i, of length n, such that $E_i = E_{i+n/I} \otimes k$.*

In the smooth case, this conjecture is trivial. For $\mathbf{P}^2$, we have $K = \mathcal{O}(-3)$, $I = 3$ and $k = \mathcal{O}(-1)$. The helix $(\mathcal{O}(i))$ is invariant under tensoring by $\mathcal{O}(-1)$. For $\mathbf{P}^1 \times \mathbf{P}^1$, we have $K = \mathcal{O}(-2,-2)$, $I = 2$ and $k = \mathcal{O}(-1,-1)$. The helix (2.22) is invariant under tensoring with $\mathcal{O}(-1,-1)$. We should note that in both cases there exist other helices that do not respect this action. For example, on $\mathbf{P}^1 \times \mathbf{P}^1$, the helix generated by the exceptional collection $(\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1))$ is not invariant. The quiver obtained from this collection is not, then, an orbifold although it is a perfectly legitimate description of the gauge theory on $K_{\mathbf{P}^1\times\mathbf{P}^1}$. The two quivers should be related by Seiberg duality.

We can provide some evidence for this conjecture in a more general context by the following. Consider an exceptional invertible sheaf (line bundle), A. Then, clearly, $\mathrm{Ext}^i(A,A) \cong \mathrm{Ext}^i(A\otimes k^a, A\otimes k^a) = 0$ for all a and $i \neq 0$. In addition, we have $\mathrm{Ext}^i(A\otimes k^{-a}, A) = H^i(k^a)$. Now, k^a is a negative line bundle, so this vanishes for $i < 2$ by the Kodaira vanishing theorem. Furthermore, we can apply Serre duality and, using the fact that $k^{-a}\otimes K_X = k^{I-a}$ is also negative for $a < I$, we see that this vanishes for $i = 2$ also. Finally, $\mathrm{Ext}^i(A, A\otimes k^{-a}) = H^i(k^{-a})$. Since $k^{-a} = k^{-I-a} \otimes K_x$ and k^{-I-a} is positive, this Ext group also vanishes for $i > 0$. From this, we can conclude that $(A, A\otimes k^{-1}, \ldots, A\otimes k^{-I+1})$ is a strong exceptional collection.

Given that the number of elements in a full exceptional collection is equal to the rank of the Grothendieck group, this conjecture implies that the Fano index divides this rank. Again, this is trivially true in the smooth case. We hope that something like this holds in the orbifold case.

Now we can see how to undo the orbifold of the quiver. The action of k on the helix gives rise to an action on the quiver. In particular, the nodes of the quiver correspond to

Figure 5: The undone orbifold quiver for $\mathbf{P}^2$.

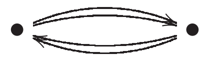

Figure 6: The undone quiver for $\mathbf{P}^1 \times \mathbf{P}^1$.

the elements in the exceptional collection. Thus, in the case of $\mathbf{P}^2$, the action is given by rotating figure 3 by 120 degrees. For $\mathbf{P}^1 \times \mathbf{P}^1$, figure 4 is rotated 180 degrees. By restricting to the invariant parts, we find the quivers in figures 5 and 6. We will make this precise in section 6.3.

For $\mathbf{P}^2$, the superpotential (2.13) is invariant under the action and thus descends to the undone orbifold. As the cone $\mathcal{O}(-1)$ over $\mathbf{P}^2$ is just $\mathbb{C}^3$ blown up at the origin, we should obtain $\mathcal{N} = 4$ SYM. To see that we do, notice that three lines in the quiver correspond to the three chiral multiplets in $\mathcal{N} = 4$ SYM. The superpotential gives the usual $[X_i, X_j]^2$ term in the action.

For $\mathbf{P}^1 \times \mathbf{P}^1$, undoing the orbifold identifies a with c and b with d in (2.14). This gives us the following superpotential

$$W = \epsilon^{ik}\epsilon^{lj} a_l b_i a_j b_k \ . \tag{3.1}$$

The cone $\mathcal{O}(-1,-1)$ over $\mathbf{P}^1 \times \mathbf{P}^1$ is a resolution of the cone cut out by $x^2+y^2+z^2+w^2 = 0$ in $\mathbb{C}^4$. This cone, termed the conifold, was first investigated in the context of AdS/CFT in [27,28]. The quiver and superpotential obtained here are exactly those derived there.

4. Orbifolds of $\mathbb{C}^*$ bundles

In this section, we will show how the above procedure can be implemented in terms of orbifolds. D-branes on an orbifold, Y/G for G some finite group, are described by the equivariant derived category, $\mathcal{D}^b_G(Y)$, [16]. For simplicity, we will assume G is Abelian. Then, this category admits an action by the group G, the quantum symmetry, and if we look at the orbits of the action, we recover the derived category of the original space $\mathcal{D}^b(Y)$. This is the essence of the procedure in the previous section.

There is a barrier to implementing this proposal, however. Given a line bundle L over Y, the total space L^n is *not* a $\mathbb{Z}_n$ orbifold of L. In this situation, $\mathbb{Z}_n$ acts on the fibers by multiplication by $e^{2i\pi/n}$, but this action preserves the origin. Thus, the quotient space is singular in codimension one. Later, we will use the McKay correspondence of [29] to overcome this difficulty, but before tackling that proposition, let us warm up by simply removing the origin and replacing our line bundles with $\mathbb{C}^*$ bundles.

$\mathbb{C}^*$ bundles are particularly relevant for AdS/CFT because the 5-fold, denoted M_5 above, is the total space of a circle bundle with Euler class, e. This bundle is a deformation retract of the total space of the $\mathbb{C}^*$ bundle also characterized by the two-form e.

In order to postpone the introduction of equivariant derived categories, we will deal instead with equivariant fiber bundles which are ordinary bundles along with a lift of the orbifold action on the base to the total space of the bundle. For $x, gx \in Y$ and $g \in G$,

this gives an identification of the points in the fibers F_x and F_{gx}. If the fiber is a vector space, F, we can take a representation of G, $r : r(g) \in GL(F)$. With this in hand, we can modify the above identification to $F_x \to r(g)F_{gx}$. This gives an action of representations on equivariant vector bundles. In the case of line bundles, the one dimensional representations form a group under the tensor product that, for G abelian, is the same as G. Choosing such an identification gives an action of G on the set of equivariant line bundles. Note that this is not an automorphism of the equivariant Picard group.

Let us assume that G acts freely on Y. Then, the set of equivariant line bundles is $H^2_G(Y, \mathbb{Z}) \cong H^2(Y/G, \mathbb{Z})$. We would like to identify the above action on the second cohomology of Y/G. Let us assume that Y is simply connected. Then, it is a standard fact that $\pi_1(Y/G) \cong H_1(Y/G, \mathbb{Z}) \cong G$ as G is Abelian. The universal coefficient theorem then tells us that $H^2_{\mathrm{tors}}(Y/G, \mathbb{Z}) \cong G$. Thus, we have a G action on $H^2(Y/G, \mathbb{Z})$ given by addition of its torsion elements which we have identified with G. These torsion elements correspond to certain line bundles over Y, and the action is simply the usual tensor product of line bundles.

Now, let us specialize to the case at hand where Y is the total space of a $\mathbb{C}^*$ bundle over X. This $\mathbb{C}^*$ bundle is characterized by its Euler class e. There is also a corresponding line bundle over X, which we denote as E, which has first Chern class e. Let π be the projection from Y to X. Then, the pullback bundle π^*E is trivial. This can be see by constructing a global nonzero section. This implies that $\pi^*(e) = 0$ in cohomology. In fact, from the the Gysin sequence and the fact that X is simply connected, we have $H^2(Y, \mathbb{Z}) \cong H^2(X, \mathbb{Z})/\mathbb{Z}e$. Now, let X have Fano index I and let $E \cong \pi^*(k^I) \cong \pi^*(K_X)$. In our usual abuse of notation, we will use the same symbols for line bundles and their first Chern classes. Then we have

$$H^2(Y, Z) \cong H^2(X, Z)/\pi^*(K_X)\mathbb{Z} \tag{4.1}$$

and $I\pi^*(k) = 0 \in H^2(Y, Z)$.

For $\mathbf{P}^2$, we have $M^5 = S^5/\mathbb{Z}_3$ and $H^2(S^5/\mathbb{Z}_3, \mathbb{Z}) = \mathbb{Z}_3$. The second cohomology consists of the torsion elements $\{0, \pi^*(k), 2\pi^*(k)\}$ with $3\pi^*(k) = 0$. These correspond precisely to the line bundles that form the exceptional collection $(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$. The action on the cohomology given by adding the form $\pi^*(k)$ gives precisely the action on the nodes of the quiver in the previous section. This lifts to an autoequivalence of the derived category given by tensoring with $\mathcal{O}(-1)$.

For $\mathbf{P}^1 \times \mathbf{P}^1$, we have $M^5 = T^{11}/\mathbb{Z}_2$, and $H^2(T^{11}/\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$. If we denote by a and b the generators of $H^2(\mathbf{P}^1 \times \mathbf{P}^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, we have $2\pi^*(a + b) = 0$ on $T^{11}/\mathbb{Z}_2$. Thus, the action on cohomology is given by the addition of $\pi^*(a+b)$, or tensoring with $\mathcal{O}(-1, -1)$. Again, this gives the previously obtained action on the exceptional collection (2.8).

5. The McKay correspondence

We would now like to be able to extend this discussion to the total space of the *line* bundle K_X over X. The goal is to obtain an autoequivalence $\mathcal{G}$ of $\mathcal{D}^b(K_X)$ such that $\mathcal{G}^I = \mathrm{id}$ where I is, as usual, the Fano index of X. As stated above, the problem is that the $\mathbb{Z}_I$ orbifold of the total space of k is *not* K_X. However, as we will see, it is possible to resolve the (conical) singularity of $k/\mathbb{Z}_I$ to obtain K_X. This will allow us, through a theorem of Bridgeland, King and Reid [29], to identify $\mathcal{D}^b_{\mathbb{Z}_I}(k)$ with $\mathcal{D}^b(K_X)$. Unfortunately, we have not been able

to use this theorem to obtain the proper autoequivalence. In the next section, we will use some further results of Bridgeland [5] to conjecturally identify this autoequivalence which corresponds to the quantum symmetry of the orbifold.

Let us briefly introduce the equivariant derived category. For further details, see [29]. Let Y be a space with a G action. An equivariant sheaf on Y is a sheaf, $\mathcal{F}$ on Y, along with a set of maps $f_g : \mathcal{F} \to g^*\mathcal{F}$ that satisfy $f_{hg} = g^*(f_h)f_g$ and $f_1 = id$. There is an action of G on $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ for $\mathcal{F}$ and $\mathcal{G}$ equivariant sheaves. Restricting to the invariant part, we obtain G-$\mathrm{Hom}(\mathcal{F}, \mathcal{G})$. From this, one can obtain the Abelian category $Coh^G(Y)$ and the usual derived categories including $\mathcal{D}^b_G(Y)$. All the usual functors such as G-Ext and pushforwards and pullbacks for equivariant maps exist and satisfy the usual relations. As with the equivariant line bundles above, there is also an action of a representation of G on an equivariant sheaf by tensor product.

In order to state the McKay correspondence of [29] we need one final ingredient, the Hilbert scheme of G-clusters on Y, G-Hilb Y. In general, this is a complicated object which we will not define. Broadly speaking, however, we can look at the space of subschemes (think submanifolds) of some space, Y. For example, we can look at the space of sets of n points in Y where the points are considered indistinguishable. This is, in general, a singular space when two points come together. The Hilbert scheme is a crepant resolution of the singular points of this space. The Hilbert scheme of G-clusters, then, is a resolution of the space of G-clusters which are, in essence, orbits of the G action on Y. Thus, G-Hilb Y is a resolution of Y/G.

With all this in hand, Bridgeland, King and Reid proved that

$$\mathcal{D}_G(Y) \cong \mathcal{D}(G\text{-Hilb } Y) \ . \tag{5.1}$$

More properly, we should replace G-Hilb Y by its irreducible component containing the free orbits. There are also some further technicalities which we will ignore.

We can now specialize to our situation. Let X and I be as before, and let $Y = k$ with the $G = \mathbb{Z}_I$ action given by rotation on the fibers. We would like to determine G-Hilb Y. The first thing to note is that, as the action of the group preserves the fiber, we can reduce the question to determining G-Hilb $\mathbb{C}$. This can be embedded in $\mathrm{Hilb}^I\,\mathbb{C}$. However, Hilbert schemes of points in dimension one are known to be isomorphic to the symmetric product $\mathbb{C}^I/S_I$ which is nonsingular in this case. Elements in this space can be written as sums of complex numbers, *i.e.,* $a + 2b + c \in \mathrm{Hilb}^4\,\mathbb{C}$. The action of G on $\mathbb{C}$ lifts to an action on $\mathbb{C}^I/S_I$, and G-Hilb is embedded as the G invariants points. It is not hard to see that these are given by $\sum_{i=0}^{I} g_i a$ for $a \in \mathbb{C}$ and g_i ranging over all the elements in G. This gives a map from $\mathbb{C}$ to G-Hilb $\mathbb{C}$. Since, off the zero section, the space Y/G is nonsingular and isomorphic to the total space K_X off the zero section, we see that $\mathbb{Z}_I$-Hilb $k \cong K_X$. Thus, we have the hoped for automorphism

$$\mathcal{D}_{\mathbb{Z}_I}(k) \cong \mathcal{D}(K_X) \ . \tag{5.2}$$

This implies that there should be an action of the group $\mathbb{Z}_I$ on the triangulated category $\mathcal{D}(K_X)$, but it appears to be difficult to determine this directly from the work of Bridgeland, King and Reid. In the next section, we will conjecture the correct autoequivalence. As we can embed the $\mathbb{C}^*$ bundles of the previous section into the line bundles of this section, we expect that, off the zero section, this should reduce to the autoequivalence discovered there.

6. The orbifold monodromy

6.1. Spherical objects and twist functors

In order to formulate our conjecture about the orbifold monodromy, we must introduce the notions of spherical objects and twist functors due to Seidel and Thomas [22]. Let Z be an algebraic variety of dimension n. An object in $S \in \mathcal{D}(Z)$ is called *spherical* if

$$\mathrm{Hom}(S, S[k]) = \begin{cases} \mathbb{C} & \text{if } k = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

An easy source of spherical objects follows from the relation (2.10). This implies that, for any exceptional object, $E \in \mathcal{D}(X)$, $s_*(E) \in \mathcal{D}(K_X)$ is spherical where s is the zero section of K_X. Given an exceptional collection, we will associate a collection of spherical objects as the pushforwards of the dual collection. Recall that these correspond to the simple modules of the completed quivers.

Seidel and Thomas then define a *twist functor* $\mathcal{T}_S$ which is the autoequivalence of the derived category which completes the following triangle for all $F \in \mathcal{D}(Z)$

$$\mathrm{Hom}(S, F) \otimes S \to F \to \mathcal{T}_S(F)\ . \tag{6.2}$$

Given a collection of spherical objects, one can define an action of the braid group by these autoequivalences of the derived category. This is further investigated in [5].

The twist functor can be written in terms of a Fourier-Mukai transform[12] as follows. Recall that for an autoequivalence, the Fourier-Mukai transform is specified by its kernel, an element in $\mathcal{D}(Z \times Z)$. Let $S \in \mathcal{D}(Z)$ be spherical. Then, the kernel that gives the twist functor is

$$S^\vee \boxtimes S \to \mathcal{O}_{\Delta Z} \tag{6.3}$$

where $S^\vee = \mathbb{R}Hom(S, \mathcal{O}_Z)$ is the dual object to S in the derived category. Twist functors generally appear in physics as monodromies around conifold points in the moduli space. For more information, see [16] and references therein.

6.2. The conjectured monodromy

We will need the following result of Bridgeland, a slight variation of Proposition 4.9 of [5][13]. Let $(E_0, \ldots, E_{n-1})$ be a simple collection in $\mathcal{D}(X)$, let E_i, $i \in \mathbb{Z}$ be the corresponding helix, and let $(S_0, \ldots, S_{n-1})$ be the corresponding spherical objects. Then, the spherical objects for the simple collection $(E_{-1}, E_0, \ldots, E_{n-2})$ are $\mathcal{T}_{S_{n-1}}(S_{n-1}, S_0, \ldots, S_{n-2})$ where the twist functor acts separately on each element in the set.

In fact, Bridgeland's proof is only given for the case dim $X = n$. However, the proof can be seen to hold in general using the definition of a helix given in section 2.3.

Now, let us define the autoequivalence,

$$\mathcal{G} = \mathcal{T}_{S_{n/I-1}} \ldots \mathcal{T}_{S_0} \mathrm{Ten}_{k^{-1}} \tag{6.4}$$

[12] For an introduction to Fourier-Mukai transforms, see [16,30].

[13] For a more careful exposition, please see the original paper.

where we act from right to left and $\mathrm{Ten}_{k^{-1}}$ denotes the autoequivalence given by tensoring with the invertible sheaf/line bundle k^{-1}. Notice that, off the zero section, this is precisely the inverse of autoequivalence found for the $\mathbb{C}^*$ bundles of section 4. With a little work, it can be seen to follow from Bridgeland's theorem that this autoequivalence gives a cyclic shift in the simple objects. In other words, it takes the collection $(S_0, \ldots, S_{n-1})$ to $(S_{n/I}, \ldots, S_{n-1}, S_0, \ldots, S_{n/I-1})$.

Physically, we know that these spherical objects correspond to fractional branes, and orbifold monodromies permute the simple branes. Furthermore, the Ith power of our autoequivalence preserves the simples. This motivates us to

Conjecture 2 *The autoequivalence of $\mathcal{D}(K_X)$ that corresponds to the action of $\mathbb{Z}_I$ on $\mathcal{D}_{\mathbb{Z}_I}(k)$ is given by the functor $\mathcal{G}$ in equation (6.4). Furthermore, this corresponds to the monodromy about the orbifold point in the moduli space for K_X.*

The second point needs some elaboration. In fact, in a multiparameter moduli space such as that for $\mathbf{P}^1 \times \mathbf{P}^1$, the orbifold point is of codimension greater than one, so there is no obvious notion of an orbifold monodromy. One can, however, take a curve in the moduli space which intersects with the orbifold point and take the monodromy constrained to that curve. This may be the proper interpretation of the monodromy obtained here.

Now, let us apply this to our two examples. For $\mathbf{P}^2$, we have

$$\mathcal{G} = \mathcal{T}_{s_*(\mathcal{O})} \mathrm{Ten}_{\mathcal{O}(1)} \tag{6.5}$$

where, by $\mathcal{O}(n)$, we mean the pullback sheaf $\pi^*(\mathcal{O}_{\mathbf{P}^2}(n))$. This is precisely the well-known orbifold monodromy for $\mathbb{C}^3/\mathbb{Z}_3$ (see, for example [16]). For $\mathbf{P}^1 \times \mathbf{P}^1$, we obtain

$$\mathcal{G} = \mathcal{T}_{s_*(\mathcal{O}(-1,0))} \mathcal{T}_{s_*(\mathcal{O})} \mathrm{Ten}_{\mathcal{O}(1,1)} \ . \tag{6.6}$$

We show in the appendix that this squares to the identity.

6.3. Obtaining the quiver

With the action of the orbifold group in hand, we can finally make rigorous the manipulations of section 3. Recall that the quiver can be determined from the Ext groups between its simple representations (2.9). We will use our knowledge of the G-Ext groups in the equivariant derived category to obtain those in the original derived category, thus determining the undone quiver.

The needed relationship is Lemma 4.1 of [29] which we quote here. Let E and F be G-sheaves on X. Then, as a representation of G, we have a direct sum decomposition

$$\mathrm{Hom}_X(E, F) = \bigoplus_{i=0}^{k} G\text{-}\mathrm{Hom}_X(E \otimes \rho_i, F) \otimes \rho_i \tag{6.7}$$

over the irreducible representations of G, $\{\rho_0, \ldots, \rho_k\}$. It is straightforward to see that this holds for Ext groups as well.

For us, G is the cyclic group $\mathbb{Z}_I$, and the representation ring is generated by a single representation. We identify the action of the generating representation with the autoequivalence conjectured above which permutes the simples. This choice is, of course, not unique. We can now decompose the simples into orbits under the permutation and, choosing one representative from each orbit, compute the Ext groups using (6.7).

For example, for $\mathbf{P}^2$, we have three simple representations S_i upon which the group $\mathbb{Z}_3$ acts as $S_i \to S_{i+1 \mod 3}$. There is only one orbit, and we must only compute one set of Ext groups:

$$\mathrm{Ext}^i(S, S) = G\text{-}\mathrm{Ext}^i(S_0, S_0) \oplus G\text{-}\mathrm{Ext}^i(S_0, S_1) \oplus G\text{-}\mathrm{Ext}^i(S_0, S_2)\ . \tag{6.8}$$

By the McKay correspondence, we can identify the equivariant Exts with the Exts of the quiver $K_{\mathbf{P}^2}$ in figure 3. The three Ext^1s from S_0 to S_1 give the three arrows in the quiver of figure 5. The three Ext^2s from S_0 to S_2 and the Ext^3 from S_0 to S_0 give the relations.

Following the same procedure for $\mathbf{P}^1 \times \mathbf{P}^1$ yields the quiver of figure 6.

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A. Squaring the autoequivalence for $\mathbf{P}^1 \times \mathbf{P}^1$

In this section, we will square the autoequivalence (6.6). The calculation uses similar techniques to those in [16], section 7.3.5. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $Y = K_X$ with projection π and zero section s as before. Let $\mathcal{O} = s_*(\mathcal{O}_X)$ where the latter is the structure sheaf on X. Furthermore, let $\mathcal{A}(n) = \mathcal{A} \otimes \pi^*\mathcal{O}_X(n)$ for any sheaf $\mathcal{A}$ on Y. Finally, for an object in the derived category $\mathcal{E}$, we denote its dual $\mathcal{E}^\vee = \mathbb{R}Hom(\mathcal{E}, \mathcal{O}_Y)$.

As a Fourier-Mukai transform, the autoequivalence (6.6) has the following kernel:

$$\mathcal{O}^\vee(0,1) \boxtimes \mathcal{O}(-1,0) \longrightarrow \mathcal{O}^\vee(1,1) \boxtimes \mathcal{O} \longrightarrow \mathcal{O}_\Delta(1,1) \tag{A.1}$$

where, in $\mathcal{O}_\Delta(1,1)$, we pull back by the diagonal map.

Squaring this, we obtain

$$\begin{aligned}\mathcal{O}(-1,0)^\vee \boxtimes \mathcal{O}(-1,0) \longrightarrow \mathcal{O}(0,-1)^\vee \boxtimes \mathcal{O}(0,1) \oplus \mathcal{O}(-2,0)^\vee \boxtimes \mathcal{O}& \\ \longrightarrow \mathcal{O}(-1,-1)^\vee \boxtimes \mathcal{O}(1,1) \longrightarrow \mathcal{O}_\Delta(2,2)\ .&\end{aligned} \tag{A.2}$$

We can write this as $Cone(\mathcal{A} \longrightarrow \mathcal{O}_\Delta(2,2))$ where $\mathcal{A}$ consists of the first three terms in the sequence.

Because X is the zero section of the bundle Y, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(2,2) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O} \longrightarrow 0 \ . \tag{A.3}$$

From this, we can determine that $\mathcal{O}(a,b)^\vee = \mathcal{O}(-2-a,-2-b)[-1]$. Substituting this into $\mathcal{A}$, we obtain

$$\begin{aligned}\mathcal{A}[1] = \mathcal{O}(-1,-2)\boxtimes\mathcal{O}(-1,0) \longrightarrow \mathcal{O}(-2,-1)\boxtimes\mathcal{O}(0,1)\oplus\mathcal{O}(0,-2)\boxtimes\mathcal{O} \\ \longrightarrow \mathcal{O}(-1,-1)\boxtimes\mathcal{O}(1,1)\end{aligned} \tag{A.4}$$

which is the pushforward of a sequence on X. If we conjugate with the action of tensor products with $\mathcal{O}(1,1)$ we obtain

$$\mathcal{O}(0,-1)\boxtimes\mathcal{O}(2,1)^\vee \longrightarrow \mathcal{O}(-1,0)\boxtimes\mathcal{O}(1,0)^\vee\oplus\mathcal{O}(1,-1)\boxtimes\mathcal{O}(1,1)^\vee \longrightarrow \mathcal{O}\boxtimes\mathcal{O}^\vee \tag{A.5}$$

where we have used that $\mathcal{O}(a,b)^\vee = \mathcal{O}(-a,-b)$ on X. Finally, it follows from theorem 4.4.3 of [25] that this sequence is a resolution of the diagonal. The conjugation by $\mathcal{O}(1,1)$ has no effect on the identity, giving $\mathcal{A} = \mathcal{O}_{\Delta X}[-1]$. Then, using the exact sequence (A.3) we can write the total action as

$$\mathit{Cone}\left((\mathcal{O}_{\Delta Y}(2,2)\longrightarrow\mathcal{O}_{\Delta Y})[-1]\longrightarrow\mathcal{O}_{\Delta Y}(2,2)\right) = \mathcal{O}_{\Delta Y} \tag{A.6}$$

This proves that the autoequivalence squares to the identity.

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